Advanced Macroeconomics: Mid-term Exam

Growth Theory: Solow–Swan, Diamond OLG, RCK, and Romer Models

Time Allowed: 120 minutes — Total: 100 marks

Instructions. Answer all three questions. Show all mathematical derivations clearly. Answers that state final results without derivation will receive limited credit. You may use a calculator. Unless otherwise stated, assume all variables are positive and all parameters lie in economically meaningful ranges.

Question	Core Models	Marks
Question 1	Solow–Swan, convergence, growth accounting	35
Question 2	Diamond OLG, ageing, saving, dynamic efficiency	35
Question 3	RCK and Romer, optimal saving, R&D-based growth	30
Total		**100**

Question 1: Cross-Country Income Gaps, Conditional Convergence, and Kaldor Facts [Total: 35 marks]

A central stylised fact in growth economics is that income per capita differs enormously across countries. Some economies experienced rapid catch-up growth, while others remained far behind the world technology frontier. At the same time, advanced economies display approximately stable factor shares, no obvious long-run trend in the real interest rate, and sustained growth in output per worker.

Consider a discrete-time Solow–Swan economy with Cobb–Douglas production:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where

$$A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t.$$

Capital evolves according to

$$K_{t+1} = (1-\delta)K_t + sY_t, \qquad 0 < s < 1, \qquad 0 < \delta < 1.$$

Define

$$k_t \equiv \frac{K_t}{A_t L_t}, \qquad y_t \equiv \frac{Y_t}{A_t L_t}, \qquad c_t \equiv \frac{C_t}{A_t L_t},$$

and let

$$D \equiv (1+n)(1+g), \qquad b \equiv D - (1-\delta) = n + g + ng + \delta.$$

(a) Derive the intensive-form production function and the exact law of motion:

$$k_{t+1} = \mathcal{G}(k_t).$$

Then derive $k_{t+1} - k_t$ and interpret the economic meaning of bk_t.

(b) Solve for the positive steady-state values k^*, y^*, and c^*. Then derive the comparative statics of k^* and y^* with respect to s, n, g, and δ.

(c) Suppose two countries share the same technology growth rate g and depreciation rate δ, but differ in saving and population growth:

$$(s_H, n_H) = (0.30, 0.005), \qquad (s_L, n_L) = (0.10, 0.025),$$

with

$$\alpha = \frac{1}{3}, \qquad g = 0.02, \qquad \delta = 0.05.$$

Using the exact formula for b, compute the long-run ratio

$$\frac{(Y/L)_H^*}{(Y/L)_L^*}$$

at a common technology level A_t. Interpret the result as a Solow-style explanation of cross-country income gaps.

(d) Prove conditional convergence around the steady state. Derive the local convergence coefficient

$$\lambda = \mathcal{G}'(k^*).$$

Then explain why, conditional on the same structural parameters, a poorer economy grows faster than a richer economy.

(e) Use the production function to derive the growth-accounting decomposition for output per worker:

$$\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1-\alpha)\Delta \ln A_t.$$

If an advanced economy has

$$\Delta \ln(Y_t/L_t) = 0.02, \qquad \Delta \ln(K_t/L_t) = 0.02, \qquad \alpha = \frac{1}{3},$$

infer the implied growth rate of labour-augmenting technology. Explain how this relates to the Kaldor facts.

Question 2: Ageing, Endogenous Saving, and Dynamic Efficiency in the Diamond OLG Model [Total: 35 marks]

Many advanced economies face population ageing: fertility rates have fallen, dependency ratios have risen, and governments are under pressure to reform pension systems. This question uses the Diamond OLG model to study how ageing and pay-as-you-go pensions affect saving, capital accumulation, and dynamic efficiency.

Individuals live for two periods. Preferences are

$$U(c_{1t}, c_{2,t+1}) = u(c_{1t}) + \beta u(c_{2,t+1}), \qquad 0 < \beta < 1,$$

with

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0.$$

Without pensions,

$$c_{1t} + s_t = A_t w_t, \qquad c_{2,t+1} = (1 + r_{t+1})s_t.$$

Firms operate

$$Y_t = A_t L_t f(k_t), \qquad f(k) = k^\alpha, \qquad 0 < \alpha < 1,$$

where

$$k_t = \frac{K_t}{A_t L_t}, \qquad A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t.$$

There is no capital depreciation. Let

$$D \equiv (1+n)(1+g).$$

(a) Solve the young household's saving problem. Derive the Euler equation and show that

$$s_t = \sigma(r_{t+1})A_t w_t,$$

where

$$\sigma(r) = \frac{\beta^{1/\theta}}{\beta^{1/\theta} + (1+r)^{1-1/\theta}}.$$

Then derive $\sigma'(r)$ and discuss the cases $\theta < 1$, $\theta = 1$, and $\theta > 1$.

(b) Derive the firm's first-order conditions:

$$r_t = f'(k_t), \qquad w_t = f(k_t) - f'(k_t)k_t.$$

Using market clearing,

$$L_t^D = L_t, \qquad K_{t+1} = L_t s_t,$$

derive the implicit transition equation:

$$k_{t+1} = \frac{1}{D}\sigma\big(f'(k_{t+1})\big)\left[f(k_t) - f'(k_t)k_t\right].$$

(c) Now impose log utility, $\theta = 1$. Derive

$$k_{t+1} = Bk_t^\alpha, \qquad B \equiv \frac{\beta}{1+\beta}\frac{1-\alpha}{D}.$$

Solve for k^*. Then derive the Golden Rule capital stock k_{GR} and the dynamic-efficiency condition.

(d) Consider an ageing shock that lowers population growth from n_0 to $n_1 < n_0$, holding all other parameters fixed. Derive analytically how this shock affects D, B, k^*, k_{GR}, and $r^* = f'(k^*)$. Does population ageing necessarily generate dynamic inefficiency? Explain carefully.

(e) Suppose the government introduces a pay-as-you-go pension. Each young individual pays a tax τA_t when young. When old, the individual receives

$$T_{t+1} = (1+n)\tau A_{t+1}.$$

The budget constraints become

$$c_{1t} + s_t = A_t w_t - \tau A_t,$$

$$c_{2,t+1} = (1 + r_{t+1})s_t + (1+n)\tau A_{t+1}.$$

Assume log utility. Derive the optimal saving function s_t as a function of w_t, r_{t+1}, τ, D, and A_t. Show mathematically why a positive PAYG tax reduces private saving.

Question 3: Optimal Saving, Innovation Policy, and Endogenous Growth [Total: 30 marks]

Suppose a government is concerned that capital accumulation alone cannot explain sustained growth in living standards. It considers an innovation strategy that reallocates skilled workers from final-goods production into R&D. This question compares the RCK model, where technological growth is exogenous, with the Romer model, where technological progress is produced by researchers.

Part I: RCK Economy

There is no population growth and no depreciation. Technology grows at

$$A_{t+1} = (1+g)A_t.$$

Households have preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \qquad u(c) = \frac{c^{1-\theta} - 1}{1-\theta}.$$

Production is

$$Y_t = A_t L f(k_t), \qquad f(k) = k^\alpha, \qquad k_t = \frac{K_t}{A_t L}, \qquad \widetilde{c}_t = \frac{c_t}{A_t}.$$

(a) Derive the RCK transition equations:

$$\widetilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1},$$

and

$$\left(\frac{\widetilde{c}_{t+1}}{\widetilde{c}_t}\right)^\theta = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^\theta}.$$

Then solve for k^* under Cobb–Douglas production.

(b) Derive the Golden Rule capital stock:

$$k_{\text{GR}} = \left(\frac{\alpha}{g}\right)^{1/(1-\alpha)}.$$

Under

$$\beta(1+g)^{1-\theta} < 1,$$

prove that

$$k^* < k_{\text{GR}}.$$

Give the economic intuition.

(c) Rewrite the RCK model as a two-dimensional first-order system in $(k_t, \widetilde{c}_t)$. Derive the $\Delta k = 0$ and $\Delta\widetilde{c} = 0$ loci. Explain why the steady state is saddle-path stable and why the transversality condition matters.

Part II: Romer Economy

Now suppose technology is endogenous:

$$Y_t = A_t K_t^\alpha L_{y,t}^{1-\alpha}, \qquad A_{t+1} - A_t = \bar{z} A_t L_{a,t}.$$

Total labour is fixed:

$$\bar{L}_t = \bar{L}, \qquad L_{a,t} = \bar{\ell}\bar{L}, \qquad L_{y,t} = (1-\bar{\ell})\bar{L}.$$

Capital evolves as

$$K_{t+1} - K_t = sY_t - \delta K_t.$$

(d) Derive the growth rate of ideas,

$$g_A = \frac{A_{t+1} - A_t}{A_t}.$$

Then prove that

$$g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1-\alpha}.$$

Explain why the Romer model can generate sustained per-capita growth even when population is constant.

(e) Suppose

$$\alpha = \frac{1}{3}, \qquad \bar{z} = 0.0005, \qquad \bar{L} = 1000.$$

The government raises the research-labour share from

$$\bar{\ell}_0 = 0.02$$

to

$$\bar{\ell}_1 = 0.05.$$

Compute the old and new values of g_A and g_y. Then compute the immediate effect on output per person, holding A_t and $K_t/\bar{L}$ fixed:

$$\frac{y_t(\bar{\ell}_1)}{y_t(\bar{\ell}_0)}.$$

Interpret the short-run cost and long-run benefit of the innovation policy.